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# Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature

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**Abstract** The Wintgen inequality (1979) is a sharp geometric inequality for surfaces in the 4-dimensional Euclidean space involving the Gauss curvature (intrinsic invariant) and the normal curvature and squared mean curvature (extrinsic invariants), respectively. De Smet et al. (Arch. Math. (Brno) 35:115–128, 1999) conjectured a generalized Wintgen inequality for submanifolds of arbitrary dimension and codimension in Riemannian space forms. This conjecture was proved by Lu (J. Funct. Anal. 261:1284–1308, 2011) and by Ge and Tang (Pac. J. Math. 237:87–95, 2008), independently. In the present paper we establish a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.

**Keywords** Wintgen inequality · Statistical manifold · Statistical submanifold · Dual connections

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# 1 Introduction

For surfaces $M^2$ of the Euclidean space $\mathbb{E}^3$, the Euler inequality $G \leq \|H\|^2$ is fulfilled, where $G$ is the (intrinsic) Gauss curvature of $M^2$ and $\|H\|^2$ is the (extrinsic) squared mean curvature of $M^2$.

Furthermore, $G = \|H\|^2$ everywhere on $M^2$ if and only if $M^2$ is totally umbilical, or still, by a theorem of Meusnier, if and only if $M^2$ is (a part of) a plane $\mathbb{E}^2$ or, it is (a part of) a round sphere $S^2$ in $\mathbb{E}^3$.

In 1979, Wintgen [25] proved that the Gauss curvature $G$, the squared mean curvature $\|H\|^2$ and the normal curvature $G^\perp$ of any surface $M^2$ in $\mathbb{E}^4$ always satisfy the inequality

$$G \leq \|H\|^2 - |G^\perp|;$$

the equality holds if and only if the ellipse of curvature of $M^2$ in $\mathbb{E}^4$ is a circle.

The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically.

A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [5].

Later, the Wintgen inequality was extended by Rouxel [20] and by Guadalupe and Rodriguez [10] independently, for surfaces $M^2$ of arbitrary codimension $m$ in real space forms $\widetilde{M}^{2+m}(c)$; namely

$$G \leq \|H\|^2 - |G^\perp| + c.$$

The equality case was also investigated.

A corresponding inequality for totally real surfaces in $n$-dimensional complex space forms was obtained in [13]. The equality case was studied and a non-trivial example of a totally real surface satisfying the equality case identically was given.

In 1999, De Smet et al. [7] formulated the conjecture on Wintgen inequality for submanifolds of real space forms, which is also known as the *DDVV conjecture*.

This conjecture was proven by the authors for submanifolds $M^n$ of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature $c$.

Recently, the DDVV conjecture was finally settled for the general case by Lu [12] and independently by Ge and Tang [9].

One of the present authors obtained generalized Wintgen inequalities for Lagrangian submanifolds in complex space forms [14] and Legendrian submanifolds in Sasakian space forms [15], respectively. Moreover, two of the present authors established in [3] a version of the Euler inequality and the Wintgen inequality for statistical surfaces in statistical manifolds of constant curvature.

In this paper, using the sectional curvature defined in [19], we derive a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.

## 2 Statistical manifolds and their submanifolds

A *statistical manifold* is a Riemannian manifold $(\tilde{M}^{n+k}, \tilde{g})$ of dimension $(n+k)$, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}(X, Y) = \tilde{g}(\tilde{\nabla}_Z X, Y) + \tilde{g}(X, \tilde{\nabla}^*_Z Y), \tag{2.1}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called *dual connections* (see [1,17,22]), and it is easily shown that $(\tilde{\nabla}^*)^* = \tilde{\nabla}$. The pair $(\tilde{\nabla}, \tilde{g})$ is said to be a *statistical structure*. If $(\tilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}^{n+k}$, so is $(\tilde{\nabla}^*, \tilde{g})$ [1,24].

On the other hand, any torsion-free affine connection $\tilde{\nabla}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{2.2}$$

where $\tilde{\nabla}^0$ is Levi-Civita connection on $\tilde{M}^{n+k}$.

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

A statistical structure $(\tilde{\nabla}, \tilde{g})$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$\tilde{R}(X, Y)Z = c\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\}. \tag{2.3}$$

A statistical structure $(\tilde{\nabla}, \tilde{g})$ of constant curvature 0 is called a *Hessian structure.*

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of dual connections satisfy

$$\tilde{g}(\tilde{R}^*(X, Y)Z, W) = -\tilde{g}(Z, \tilde{R}(X, Y)W). \tag{2.4}$$

From (2.4) it follows immediately that if $(\tilde{\nabla}, \tilde{g})$ is a statistical structure of constant curvature $c$, then $(\tilde{\nabla}^*, \tilde{g})$ is also a statistical structure of constant curvature $c$. In particular, if $(\tilde{\nabla}, \tilde{g})$ is Hessian, so is $(\tilde{\nabla}^*, \tilde{g})$ [8].

On a Hessian manifold $(\tilde{M}^{n+k}, \tilde{\nabla})$, let $\gamma = \tilde{\nabla}^0 - \tilde{\nabla}$. The tensor field $Q$ of type (1,3) defined by the covariant differential $Q = \tilde{\nabla}\gamma$ of $\gamma$ is said to be the *Hessian curvature tensor* for $\tilde{\nabla}$ (see [21]).

By using the Hessian curvature tensor $Q$, a Hessian sectional curvature can be defined on a Hessian manifold.

A Hessian manifold has constant Hessian sectional curvature $\tilde{c}$ if and only if (see [21])

$$Q(X, Y, Z, W) = \frac{\tilde{c}}{2}[g(X, Y)g(Z, W) + g(X, W)g(Y, Z)],$$

for all vector fields on $\tilde{M}^{n+k}$.

If $(\tilde{M}^{n+k}, \tilde{g})$ is a statistical manifold and $M^n$ a submanifold of dimension $n$ of $\tilde{M}^{n+k}$, then $(M^n, g)$ is also a statistical manifold with the induced connection by $\tilde{\nabla}$ and induced metric $g$. In the case that $(\tilde{M}^{n+k}, \tilde{g})$ is a semi-Riemannian manifold, the induced metric $g$ has to be non-degenerate. For details, see [23,24].

In the geometry of Riemannian submanifolds (see [4]), the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci.

Let denote the set of the sections of the normal bundle to $M^n$ by $\Gamma(TM^{n\perp})$.

In our case, for any $X, Y \in \Gamma(TM^n)$, according to [24], the corresponding Gauss formulas are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$
$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{2.6}$$

where $h, h^* : \Gamma(TM^n) \times \Gamma(TM^n) \to \Gamma(TM^{n\perp})$ are symmetric and bilinear, called the *imbedding curvature tensor* of $M^n$ in $\tilde{M}^{n+k}$ for $\tilde{\nabla}$ and the *imbedding curvature tensor* of $M^n$ in $\tilde{M}^{n+k}$ for $\tilde{\nabla}^*$, respectively.

In [24], it is also proved that $(\nabla, g)$ and $(\nabla^*, g)$ are dual statistical structures on $M^n$.

Since $h$ and $h^*$ are bilinear, we have the linear transformations $A_\xi$ and $A^*_\xi$ on $TM^n$ defined by

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi), \tag{2.7}$$
$$g(A^*_\xi X, Y) = \tilde{g}(h^*(X, Y), \xi), \tag{2.8}$$

for any $\xi \in \Gamma(TM^{n\perp})$ and $X, Y \in \Gamma(TM^n)$. Further, see [24], the corresponding Weingarten formulas are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi, \tag{2.9}$$
$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi, \tag{2.10}$$

for any $\xi \in \Gamma(TM^{n\perp})$ and $X \in \Gamma(TM^n)$. The connections $\nabla^\perp_X$ and $\nabla^{*\perp}_X$ given by (2.9) and (2.10) are Riemannian dual connections with respect to induced metric on $\Gamma(TM^{n\perp})$.

Let $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_k\}$ be orthonormal tangent and normal frames, respectively, on $M^n$. Then the mean curvature vector fields are defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h\left(e_i, e_i\right) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^\alpha_{ii}\right)\xi_\alpha, \quad h^\alpha_{ij} = \tilde{g}\left(h\left(e_i, e_j\right), \xi_\alpha\right), \tag{2.11}$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*\left(e_i, e_i\right) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^{*\alpha}_{ii}\right)\xi_\alpha, \quad h^{*\alpha}_{ij} = \tilde{g}\left(h^*\left(e_i, e_j\right), \xi_\alpha\right), \tag{2.12}$$

for $1 \le i, j \le n$ and $1 \le \alpha \le k$ (see also [6]).

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.

**Proposition 2.1** [24] *Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on $\tilde{M}^{n+k}$ and $\nabla$ the induced connection by $\tilde{\nabla}$ on $M^n$. Let $\tilde{R}$ and $R$ be the Riemannian curvature tensors for $\tilde{\nabla}$ and $\nabla$, respectively. Then,*

$$\begin{aligned} \tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + \tilde{g}(h(X,Z),h^*(Y,W)) \\ &\quad -\tilde{g}(h^*(X,W),h(Y,Z)), \qquad (2.13) \\ (\tilde{R}(X,Y)Z)^\perp &= \nabla^\perp_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \\ &\quad -\{\nabla^\perp_Y h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\}, \\ \tilde{g}(R^\perp(X,Y)\xi,\eta) &= \tilde{g}(\tilde{R}(X,Y)\xi,\eta) + g([A^*_\xi,A_\eta]X,Y), \qquad (2.14) \end{aligned}$$

*where $R^\perp$ is the Riemannian curvature tensor of $\nabla^\perp$ on $TM^{n\perp}$, $\xi,\eta \in \Gamma(TM^{n\perp})$ and $[A^*_\xi,A_\eta] = A^*_\xi A_\eta - A_\eta A^*_\xi$.*

For the equations of Gauss, Codazzi and Ricci with respect to the connection $\tilde{\nabla}^*$ on $M^n$, we have

**Proposition 2.2** [24] *Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on $\tilde{M}^{n+k}$ and $\nabla^*$ the induced connection by $\tilde{\nabla}^*$ on $M^n$. Let $\tilde{R}^*$ and $R^*$ be the Riemannian curvature tensors for $\tilde{\nabla}^*$ and $\nabla^*$, respectively. Then,*

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X,Y)Z,W) &= g(R^*(X,Y)Z,W) + \tilde{g}(h^*(X,Z),h(Y,W)) \\ &\quad -\tilde{g}(h(X,W),h^*(Y,Z)), \qquad (2.15) \\ (\tilde{R}^*(X,Y)Z)^\perp &= \nabla^{*\perp}_X h^*(Y,Z) - h^*(\nabla^*_X Y,Z) - h^*(Y,\nabla^*_X Z) \\ &\quad -\{\nabla^{*\perp}_Y h^*(X,Z) - h^*(\nabla^*_Y X,Z) - h^*(X,\nabla^*_Y Z)\}, \\ \tilde{g}(R^{*\perp}(X,Y)\xi,\eta) &= \tilde{g}(\tilde{R}^*(X,Y)\xi,\eta) + g([A_\xi,A^*_\eta]X,Y), \qquad (2.16) \end{aligned}$$

*where $R^{*\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp *}$ on $TM^{n\perp}$, $\xi,\eta \in \Gamma(TM^{n\perp})$ and $[A_\xi,A^*_\eta] = A_\xi A^*_\eta - A^*_\eta A_\xi$.*

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [2].

## 3 Statistical surfaces in statistical manifolds of constant curvature

Let $(\tilde{M}^3,\tilde{g})$ be a 3-dimensional statistical manifold of constant curvature $c$ and $M^2$ a surface of $\tilde{M}$. Denote the Gauss curvature, the mean curvature and the dual mean curvature of $M$, by $G$, $H$ and $H^*$, respectively. In [3], a version of the Euler inequality for statistical surfaces was given.

**Proposition 3.1** [3] *Let $M^2$ be a surface in a 3-dimensional statistical manifold of constant curvature $c$. Then its Gauss curvature satisfies:*

$$G \leq 2\|H\| \cdot \|H^*\| - c. \qquad (3.1)$$

Some examples of statistical surfaces satisfying the equality case of the above Euler inequality can be provided by the following.

*Example 1* (*A trivial example*) Recall Lemma 5.3 of Furuhata [8].

Let $(\mathbb{H}, \tilde{\nabla}, \tilde{g})$ be a Hessian manifold of constant Hessian sectional curvature $\tilde{c} \neq 0$, $(M, \nabla, g)$ a trivial Hessian manifold and $f : M \longrightarrow \mathbb{H}$ a statistical immersion of codimension one. Then one has:

$$A^* = 0, \quad h^* = 0, \quad \|H^*\| = 0.$$

Thus, if $\dim M = 2$, the immersion $f$ of codimension one satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

*Example 2* Let $(\mathbb{H}^3, \tilde{g})$ be the upper half space of constant sectional curvature $-1$, i.e.,

$$\mathbb{H}^3 = \{y = (y^1, y^2, y^3) \in \mathbb{R}^3 : y^3 > 0\}, \quad \tilde{g} = (y^3)^{-2} \sum_{k=1}^{3} dy^k dy^k.$$

An affine connection $\tilde{\nabla}$ on $\mathbb{H}^3$ is given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y^3}} \frac{\partial}{\partial y^3} = (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij} (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^3} = \tilde{\nabla}_{\frac{\partial}{\partial y^3}} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, 2$. The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ is identically zero, i.e., $c = 0$. Thus $(\mathbb{H}^3, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold of constant Hessian sectional curvature 4 (see [21]).

Now let consider a horosphere $M^2$ in $\mathbb{H}^3$ having null Gauss curvature, i.e., $G \equiv 0$ (for details, see [11]). If $f : M^2 \longrightarrow \mathbb{H}^3$ is a statistical immersion of codimension one, then, by using Lemma 4.1 of [16], we deduce $A^* = 0$, and then $H^* = 0$. This implies that the horosphere $M^2$ satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

More generally, let consider a 4-dimensional statistical manifold of constant curvature $c$, i.e. $(\tilde{M}^4, c)$, and a surface $M^2$ of $\tilde{M}^4$. We respectively denote the Gauss curvature, the normal curvature and the Gauss curvature with respect to the Levi-Civita connection by $G$, $G^\perp$ and $G^0$. Similarly, we respectively denote the mean vector field, the dual mean curvature and the sectional curvature with respect to the Levi-Civita connection by $H$, $H^*$ and $\tilde{K}^0$. We have the following Wintgen inequalities.

**Theorem 3.2** [3] *Let $M^2$ be a statistical surface in a 4-dimensional statistical manifold $(\tilde{M}^4, c)$ of constant curvature $c$. Then*

$$G + |G^\perp| + 2G^0 \leq \frac{1}{2}(\|H\|^2 + \|H^*\|^2) - c + 2\tilde{K}^0(e_1 \wedge e_2).$$

*In particular, for $c = 0$ we derive the following.*

**Corollary 3.3** [3] *Let $M^2$ be a statistical surface of a Hessian 4-dimensional statistical manifold $\tilde{M}^4$ of Hessian curvature 0. Then*:

$$G+|G^{\perp}|+2G^{0}\leq\frac{1}{2}(\|H\|^{2}+\|H^{*}\|^{2}).$$

## 4 Wintgen inequality for statistical submanifolds

Let $M^n$ be an $n$-dimensional statistical submanifold of a $(n+m)$-dimensional statistical manifold $(\tilde{M}^{n+m}, c)$ of constant curvature $c$.

The sectional curvature $K$ on $M^n$ is defined by [3] (see also [18,19])

$$K(X\wedge Y)=\frac{1}{2}[g(R(X,Y)X,Y)+g(R^{*}(X,Y)X,Y)],$$

for any orthonormal vectors $X, Y \in T_pM^n,\ p \in M^n$.

In the case of the Levi-Civita connection, the above definition coincides (up to the sign) to the standard definition of the sectional curvature.

Let $p \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis of $T_pM^n$. Then the normalized scalar curvature $\rho$ is defined by (see [7]):

$$\begin{aligned}\rho&=\frac{2}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}K\left(e_{i}\wedge e_{j}\right)\\&=\frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}\left[g\left(R\left(e_{i},e_{j}\right)e_{i},e_{j}\right)+g\left(R^{*}\left(e_{i},e_{j}\right)e_{i},e_{j}\right)\right].\end{aligned}$$

By using the Gauss equations for the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively, we obtain

$$\begin{aligned}\rho=\frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}&\left[-c-g\left(h\left(e_{i},e_{i}\right),h^{*}\left(e_{j},e_{j}\right)\right)+g\left(h^{*}\left(e_{i},e_{j}\right),h\left(e_{i},e_{j}\right)\right)\right.\\&\left.-c-g\left(h^{*}\left(e_{i},e_{i}\right),h\left(e_{j},e_{j}\right)\right)+g\left(h\left(e_{i},e_{j}\right),h^{*}\left(e_{i},e_{j}\right)\right)\right].\end{aligned}$$

Denoting as usual by

$$\begin{gathered}h_{ij}^{r}=g\left(h\left(e_{i},e_{j}\right),\xi_{r}\right),\quad h_{ij}^{*r}=g\left(h^{*}\left(e_{i},e_{j}\right),\xi_{r}\right),\\\forall i,j=1,\dots,n\text{ and }r=1,\dots,m,\end{gathered}$$

the above equation becomes

$$\rho=-c+\frac{1}{n\left(n-1\right)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(2h_{ij}^{r}h_{ij}^{*r}-h_{ii}^{*r}h_{jj}^{r}-h_{ii}^{r}h_{jj}^{*r}\right).\tag{4.1}$$

On the other hand, the normalized normal scalar curvature $\rho^{\perp}$ is defined by (see also [3]):

$$\rho^{\perp}=\frac{1}{n\,(n-1)}\left\{\sum_{1\le r<s\le m}\sum_{1\le i<j\le n}\left[g\left(R^{\perp}\left(e_i,e_j\right)\xi_r,\xi_s\right)+g\left(R^{*\perp}\left(e_i,e_j\right)\xi_r,\xi_s\right)\right]^2\right\}^{\frac{1}{2}}.$$

The Ricci equations for the dual connections $\tilde{\nabla}$, and $\tilde{\nabla}^{*}$, respectively, imply

$$\rho^{\perp}=\frac{1}{n\,(n-1)}\left\{\sum_{1\le r<s\le m}\sum_{1\le i<j\le n}\left[g\left(\left[A^{*}_{\xi_r},A_{\xi_s}\right]e_i,e_j\right)+g\left(\left[A_{\xi_r},A^{*}_{\xi_s}\right]e_i,e_j\right)\right]^2\right\}^{\frac{1}{2}}$$

or equivalently,

$$\rho^{\perp}=\frac{1}{n\,(n-1)}\left\{\sum_{1\le r<s\le m}\sum_{1\le i<j\le n}\left[\sum_{k=1}^{n}\left(h^{s}_{ik}h^{*r}_{jk}-h^{*r}_{ik}h^{s}_{jk}+h^{*s}_{ik}h^{r}_{jk}-h^{r}_{ik}h^{*s}_{jk}\right)\right]^2\right\}^{\frac{1}{2}}.$$

It follows that

$$\rho^{\perp}=\frac{1}{n\,(n-1)}\left\{\sum_{1\le r<s\le m}\sum_{1\le i<j\le n}\left[\sum_{k=1}^{n}\left(\left(h^{s}_{ik}+h^{*s}_{ik}\right)\left(h^{r}_{jk}+h^{*r}_{jk}\right)-h^{s}_{ik}h^{r}_{jk}-h^{*s}_{ik}h^{*r}_{jk}\right.\right.\right.$$
$$\left.\left.\left.-\left(h^{r}_{ik}+h^{*r}_{ik}\right)\left(h^{s}_{jk}+h^{*s}_{jk}\right)+h^{r}_{ik}h^{s}_{jk}+h^{*r}_{ik}h^{*s}_{jk}\right)\right]^2\right\}^{\frac{1}{2}}.$$

It is known that the components of the second fundamental form $h^0$ of $M^n$ with respect to the Levi-Civita connection $\tilde{\nabla}^0$ are given by $2h^{0r}_{ik}=h^{r}_{ik}+h^{*r}_{ik}$, $\forall i,k=1,\ldots,n$, $r=1,\ldots,m$. Then we can write

$$\rho^{\perp}=\frac{1}{n\,(n-1)}\left\{\sum_{1\le r<s\le m}\sum_{1\le i<j\le n}\left[\sum_{k=1}^{n}\left(4\left(h^{0s}_{ik}h^{0r}_{jk}-h^{0r}_{ik}h^{0s}_{jk}\right)+\left(h^{r}_{ik}h^{s}_{jk}-h^{s}_{ik}h^{r}_{jk}\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(h^{*r}_{ik}h^{*s}_{jk}-h^{*s}_{ik}h^{*r}_{jk}\right)\right)\right]^2\right\}^{\frac{1}{2}}. \tag{4.2}$$

We shall use the algebraic inequality

$$(a+b+c)^2\le 3(a^2+b^2+c^2),\quad \forall a,b,c\in\mathbb{R}.$$

Therefore

$$\rho^{\perp} \leq \frac{3}{n\,(n-1)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left(16\left[\sum_{k=1}^{n}\left(h_{ik}^{0s}h_{jk}^{0r}-h_{ik}^{0r}h_{jk}^{0s}\right)\right]^{2}\right.\right.$$
$$\left.\left.+\left[\sum_{k=1}^{n}\left(h_{ik}^{r}h_{jk}^{s}-h_{ik}^{s}h_{jk}^{r}\right)\right]^{2}+\left[\sum_{k=1}^{n}\left(h_{ik}^{*r}h_{jk}^{*s}-h_{ik}^{*s}h_{jk}^{*r}\right)\right]^{2}\right)\right\}^{\frac{1}{2}}. \quad (4.3)$$

Recall an inequality from [12] (see also [14])

$$\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ii}^{r}-h_{jj}^{r}\right)^{2}+2n\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ij}^{r}\right)^{2}$$
$$\geq 2n\left[\sum_{1\leq i<j\leq n}\sum_{1\leq r<s\leq m}\left(\sum_{k=1}^{n}(h_{jk}^{r}h_{ik}^{s}-h_{ik}^{r}h_{jk}^{s})\right)^{2}\right]^{\frac{1}{2}}.$$

Similarly, we have

$$\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ii}^{*r}-h_{jj}^{*r}\right)^{2}+2n\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ij}^{*r}\right)^{2}$$
$$\geq 2n\left[\sum_{1\leq i<j\leq n}\sum_{1\leq r<s\leq m}\left(\sum_{k=1}^{n}(h_{jk}^{*r}h_{ik}^{*s}-h_{ik}^{*r}h_{jk}^{*s})\right)^{2}\right]^{\frac{1}{2}}$$

and

$$\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ii}^{0r}-h_{jj}^{0r}\right)^{2}+2n\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ij}^{0r}\right)^{2}$$
$$\geq 2n\left[\sum_{1\leq i<j\leq n}\sum_{1\leq r<s\leq m}\left(\sum_{k=1}^{n}(h_{jk}^{0r}h_{ik}^{0s}-h_{ik}^{0r}h_{jk}^{0s})\right)^{2}\right]^{\frac{1}{2}}.$$

Summing up the above three inequalities, from (4.3) we obtain

$$\rho^{\perp} \leq \frac{3}{2n^{2}\,(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[\left(h_{ii}^{r}-h_{jj}^{r}\right)^{2}+\left(h_{ii}^{*r}-h_{jj}^{*r}\right)^{2}+16\left(h_{ii}^{0r}-h_{jj}^{0r}\right)^{2}\right]$$
$$+\frac{3}{n\,(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[\left(h_{ij}^{r}\right)^{2}+\left(h_{ij}^{*r}\right)^{2}+16\left(h_{ij}^{0r}\right)^{2}\right]. \quad (4.4)$$

Also, we can write

$$n^2 \|H\|^2 = \sum_{r=1}^{m} \left( \sum_{i=1}^{n} h_{ii}^r \right)^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left( h_{ii}^r - h_{jj}^r \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^r h_{jj}^r$$

and similarly,

$$n^2 \left\| H^* \right\|^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left( h_{ii}^{*r} - h_{jj}^{*r} \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^{*r} h_{jj}^{*r}$$

and

$$n^2 \left\| H^0 \right\|^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left( h_{ii}^{0r} - h_{jj}^{0r} \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^{0r} h_{jj}^{0r}.$$

Substituting in (4.4), we get

$$\begin{aligned}
\rho^\perp \le\; & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left( h_{ii}^r h_{jj}^r + h_{ii}^{*r} h_{jj}^{*r} + 16 h_{ii}^{0r} h_{jj}^{0r} \right) \\
& + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[ \left( h_{ij}^r \right)^2 + \left( h_{ij}^{*r} \right)^2 + 16 \left( h_{ij}^{0r} \right)^2 \right] \\
=\; & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[ \left( h_{ii}^r + h_{ii}^{*r} \right) \left( h_{jj}^r + h_{jj}^{*r} \right) - h_{ii}^{*r} h_{jj}^r - h_{ii}^r h_{jj}^{*r} + 16 h_{ii}^{0r} h_{jj}^{0r} \right] \\
& + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[ \left( h_{ij}^r + h_{ij}^{*r} \right)^2 - 2 h_{ij}^r h_{ij}^{*r} + 16 \left( h_{ij}^{0r} \right)^2 \right].
\end{aligned}$$

Using again $2h_{ij}^{0r} = h_{ij}^r + h_{ij}^{*r}$, $\forall i, j = 1, \ldots, n$, $r = 1, \ldots, m$, we obtain

$$\begin{aligned}
\rho^\perp \le\; & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[ 20 h_{ii}^{0r} h_{jj}^{0r} - h_{ii}^{*r} h_{jj}^r - h_{ii}^r h_{jj}^{*r} - 20 (h_{ij}^{0r})^2 + 2 h_{ij}^r h_{ij}^{*r} \right].
\end{aligned} \tag{4.5}$$

Substituting (4.1) in (4.5), one leads to

$$\rho^{\perp} \leq \frac{3}{2}\|H\|^2 + \frac{3}{2}\left\|H^*\right\|^2 + 24\left\|H^0\right\|^2 - 3\rho - 3c - \frac{60}{n(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[h_{ii}^{0r}h_{jj}^{0r} - \left(h_{ij}^{0r}\right)^2\right]. \tag{4.6}$$

If we denote by

$$\tilde{\rho}^0 = \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n}\tilde{R}^0\left(e_i, e_j, e_i, e_j\right),$$

the Gauss equation for the Levi-Civita connection $\tilde{\nabla}^0$ gives

$$\tilde{\rho}^0 = \rho^0 - \frac{2}{n(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[h_{ii}^{0r}h_{jj}^{0r} - \left(h_{ij}^{0r}\right)^2\right]. \tag{4.7}$$

From (4.6) and (4.7) we obtain

$$\rho^{\perp} \leq \frac{3}{2}\|H\|^2 + \frac{3}{2}\left\|H^*\right\|^2 + 24\left\|H^0\right\|^2 - 3\rho - 3c + 30\left(\tilde{\rho}^0 - \rho^0\right).$$

Summarizing, we proved the following generalized Wintgen inequality.

**Theorem 4.1** *Let $M^n$ be a submanifold in a statistical manifold $(\tilde{M}^{n+m}, c)$ of constant curvature $c$. Then*

$$\rho^{\perp} + 3\rho \leq \frac{15}{2}\|H\|^2 + \frac{15}{2}\|H^*\|^2 + 12g(H, H^*) - 3c + 30(\tilde{\rho}^0 - \rho^0).$$

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## References

1. Amari, S.: Differential-Geometrical Methods in Statistics. Springer, Berlin (1985)
2. Aydin, M.E., Mihai, A., Mihai, I.: Some inequalities on submanifolds in statistical manifolds of constant curvature. Filomat **29**(3), 465–477 (2015)
3. Aydin, M.E., Mihai, I.: Wintgen inequality for statistical surfaces (2015). arXiv:1511.04987 [math.DG]
4. Chen, B.Y.: Geometry of Submanifolds. M. Dekker, New York (1973)
5. Chen, B.Y.: On Wintgen ideal surfaces. In: Mihai, A., Mihai, I. (eds.) Riemannian Geometry and Applications—Proceedings RIGA 2011, pp. 59–74. Ed. Univ. Bucureşti, Bucharest (2011)

6. Chen, B.Y.: Mean curvature and shape operator of isometric immersions in real-space-forms. Glasg. Math. J. **38**, 87–97 (1996)
7. De Smet, P.J., Dillen, F., Verstraelen, L., Vrancken, L.: A pointwise inequality in submanifold theory. Arch. Math. (Brno) **35**, 115–128 (1999)
8. Furuhata, H.: Hypersurfaces in statistical manifolds. Differ. Geom. Appl. **27**, 420–429 (2009)
9. Ge, J., Tang, Z.: A proof of the DDVV conjecture and its equality case. Pac. J. Math. **237**, 87–95 (2008)
10. Guadalupe, I.V., Rodriguez, L.: Normal curvature of surfaces in space forms. Pac. J. Math. **106**, 95–103 (1983)
11. Lopez, R.: Parabolic surfaces in hyperbolic space with constant Gaussian curvature. Bull. Belg. Math. Soc. Simon Stevin **16**, 337–349 (2009)
12. Lu, Z.: Normal scalar curvature conjecture and its applications. J. Funct. Anal. **261**, 1284–1308 (2011)
13. Mihai, A.: An inequality for totally real surfaces in complex space forms. Kragujev. J. Math. **26**, 83–88 (2004)
14. Mihai, I.: On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms. Nonlinear Anal. **95**, 714–720 (2014)
15. Mihai, I.: On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms. Tohoku Math. J. **69**(1) (2017) (to appear)
16. Min, C.R., Choe, S.O., An, Y.H.: Statistical immersions between statistical manifolds of constant curvature. Glob. J. Adv. Res. Class. Mod. Geom. **3**, 66–75 (2014)
17. Nomizu, K., Sasaki, T.: Affine Differential Geometry. Cambridge University Press, Cambridge (1994)
18. Opozda, B.: A sectional curvature for statistical structures. Linear Algebra Appl. **497**, 134–161 (2016)
19. Opozda, B.: Bochner's technique for statistical structures. Ann. Global Anal. Geom. **48**, 357–395 (2015)
20. Rouxel, B.: Sur une famille des A-surfaces d'un espace Euclidien $E^4$, p. 185. Österreischer Mathematiker Kongress, Insbruck (1981)
21. Shima, H.: The Geometry of Hessian Structures. World Scientific, Singapore (2007)
22. Simon, U.: Affine differential geometry. In: Dillen, F., Verstraelen, L. (eds.) Handbook of Differential Geometry, vol. I, pp. 905–961. North-Holland, Amsterdam (2000)
23. Uohashi, K., Ohara, A., Fujii, T.: 1-conformally flat statistical submanifolds. Osaka J. Math. **37**, 501–507 (2000)
24. Vos, P.W.: Fundamental equations for statistical submanifolds with applications to the Bartlett correction. Ann. Inst. Stat. Math. **41**(3), 429–450 (1989)
25. Wintgen, P.: Sur l'inégalité de Chen-Willmore. C.R. Acad. Sci. Paris Sér.A–B **288**, A993–A995 (1979)